THE GEOMETRY AND CONSERVATION LAWS OF SCALAR PARABOLIC SYSTEMS.

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WARM-UP

In this talk I want to consider 2nd-order PDE given by one equation for a single function of $n+1$ variables[1]. That is, I am interested in functions

$$u\colon\ \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

that satisfy a differential equation

$$F(x, u, \partial u, \partial^2 u) = 0. \tag{1}$$

I want to describe how to consider this differential equation as a geometric object, but first I recall several facts.

First, the function u can be considered as a section of the trivial line bundle (the space of "zero-jets") over $\mathbb{R}^{n+1}$,

$$\begin{array}{c} J^0(\mathbb{R}^{n+1}, \mathbb{R}) \cong \mathbb{R}^{n+1} \times \mathbb{R} \\ {\scriptstyle u}\big\uparrow\big\downarrow \\ \mathbb{R}^{n+1} \end{array}$$

Furthermore, there is a bundle of 2-jets of $J^2(\mathbb{R}^{n+1}, R)$, and a natural way to lift u to a section of the larger bundle, the *2-jet lift* of u, denoted $j^2(u)$:

$$\begin{array}{c} J^2(\mathbb{R}^{n+1}, \mathbb{R}) \\ \big\downarrow \\ J^0(\mathbb{R}^{n+1}, \mathbb{R}) \\ {\scriptstyle u}\big\uparrow\big\downarrow \\ \mathbb{R}^{n+1} \end{array} \qquad j^2(u)\colon \mathbb{R}^{n+1} \to J^2(\mathbb{R}^{n+1}, \mathbb{R})$$

The manifold $J^2(\mathbb{R}^{n+1}, \mathbb{R})$ is isomorphic to

$$\mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathrm{Sym}^2(\mathbb{R}^{n+1}),$$

which can be given coordinates

$$(x^a, u, p_a, p_{ab}).$$

Now, a generic section of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$ will not be the 2-jet lift of any function, but there is a simple geometric test to determine which sections are. First, define the 1-forms $\theta_\varnothing, \theta_a$ by

$$\begin{aligned} \theta_\varnothing &= du - p_a\, dx^a \\ \theta_a &= dp_a - p_{ab}\, dx^b. \end{aligned}$$

(Einstein summation convention assumed.) Then the following proposition is not difficult to prove.

[1] I begin with $n+1$ independent variables because I will restrict attention to parabolic equations.

Proposition 1. *An $n+1$ dimensional submanifold $\iota\colon \Sigma \hookrightarrow J^2(\mathbb{R}^{n+1}, \mathbb{R})$ that submerses onto $\mathbb{R}^{n+1}$ (under the bundle map) is locally the graph of a section of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$. This section is the 2-jet lift of a function u if and only if*

$$\iota^*\theta_\varnothing = \iota^*\theta_a = 0.$$

In other words, if the ideal

$$\mathcal{C}_{alg} = \{\theta_\varnothing, \theta_a\}$$

pulls back to zero on Σ, then Σ is (the graph of) the 2-jet lift of a function. Notice that if $\mathcal{C}_{alg}$ vanishes on Σ, then the larger ideal

$$\mathcal{C} = \{\theta_\varnothing, \theta_a,\, d\theta_\varnothing,\, d\theta_a\}$$

does too[2]. It turns out to be more useful to consider the second kind of ideal, which is *differentially closed*.

This proposition can help us study the differential equation F. Indeed, F can be thought of as a function on $J^2(\mathbb{R}^{n+1}, \mathbb{R})$, and we may consider the zero locus of F,

$$M = F^{-1}(0).$$

The relation F is truly second order if and only if it varies non-trivially in its 'second-derivative' variables, which I assume to be true. This also ensures that the zero locus of F is a manifold. I will furthermore assume that F is real analytic.

Let $\mathcal{I}$ denote the pullback of $\mathcal{C}$ to M. Then we have the following.

Corollary 1. *An $n+1$-dimensional submanifold $\iota\colon \Sigma \hookrightarrow M$ such that Σ submerses onto $\mathbb{R}^{n+1}$ satisfies the condition*

$$\iota^*\mathcal{I} = 0$$

if and only if Σ is locally the 2-jet graph of a function u that solves the differential equation (1).

This is clear, because the condition that Σ lie in M is a restatement of the fact that u satisfies the equation $F = 0$.

Exterior Differential Systems

The previous section hopefully motivates the following definition.

Definition 1. An *exterior differential system* $(M, \mathcal{I})$ is a smooth manifold M and a graded, differentially closed ideal $\mathcal{I}$ in the ring of forms $\Omega^\bullet(M)$.

A submanifold $\iota\colon \Sigma \hookrightarrow M$ is an *integral manifold* of $(M, \mathcal{I})$ if the pullback $\iota^*\mathcal{I}$ is identically zero, or equivalently, if $\phi|_{T_x\Sigma} = 0$ for all $\phi \in \mathcal{I}$ and $x \in \Sigma$.

As in the example above, one should think of the pair $(M, \mathcal{I})$ as the data of a PDE and its integral manifolds as the graphs of solutions.

There is a very well defined theory of exterior differential systems, but except in special cases, most of the results rely on $(M, \mathcal{I})$ being real analytic[3]. This is perhaps disappointing if one is concerned with more general questions of regularity. However, for questions about things such as local invariants and conservation laws it seems sufficient to understand the real analytic case.

Example 1. For a symplectic manifold M with symplectic form ω, the maximal integral manifolds of $(M, \{\omega\})$ are the Lagrangian submanifolds.

Likewise, for a contact manifold M with contact form θ, the maximal integral manifolds of $(M, \{\theta,\, d\theta\})$ are the Legendrian submanifolds.

[2] E.g., because $\iota^*(\,d\theta_\varnothing) =\, d(\iota^*(\theta_\varnothing)) = 0$.

[3] The foundational tool is the Cauchy-Kowalevski theorem, which relies crucially on real analyticity.

One upshot of the definition is that the expression of the PDE (1) as an EDS is independent of a choice of coordinates. More precisely, exterior differential systems form a category, where the most important morphisms are given by the following.

Definition 2. An *equivalence* of exterior differential systems $(M, \mathcal{I})$ and $(M', \mathcal{I}')$ is a diffeomorphism $f\colon M \to M'$ for which $f^*\mathcal{I}' = \mathcal{I}$.

The coordinate indepence of exterior differential systems is evinced by the following diagram:

(2)
$$\begin{array}{ccc} F(x^a, u, p_a, p_{ab}) & \overset{\text{EDS `functor'}}{\rightsquigarrow} & (M, \mathcal{I}) \\ \Big\downarrow{\scriptstyle \text{Change of Variables}} & & \Big\downarrow{\scriptstyle \text{EDS equivalence}} \\ \tilde{F}(\tilde{x}^a, \tilde{u}, \tilde{p}_a, \tilde{p}_{ab}) & \overset{\text{EDS `functor'}}{\rightsquigarrow} & (\tilde{M}, \tilde{\mathcal{I}}) \end{array}$$

One immediate consequence of definition 2 is that an EDS equivalence between M and $\tilde{M}$ will push integral manifolds of M forward to integral manifolds of $\tilde{M}$. In other words, an EDS equivalence preserves the structure of solutions, just as a change of variables does.

In the theory of PDE it is natural to ask the following.

Question 1. *Are two given partial differential equations F and $\tilde{F}$ related by a change of variables?*

By the 'functoriality' of diagram 2, this leads naturally to the following question, whose answer would essentially resolve the previous question.

Question 2. *When are two given exterior differential systems related by an EDS equivalence?*

Fortunately, the latter question can be studied with geometric tools.

GEOMETRY OF EXTERIOR DIFFERENTIAL SYSTEMS

An exterior differential system $(M, \mathcal{I})$ is a smooth manifold with extra structure—the ideal $\mathcal{I}$, which contains the information of solutions. Many classical geometries are characterized as manifolds with extra structure, such as Riemannian geometry, complex geometry, and symplectic geometry. From this perspective, EDS equivalences are exactly the geometry-preserving maps, just like isometries, bi-holomorphisms, and symplectomorphism in their respective geometries.

Cartan's method of equivalence is a classical tool that is useful in studying geometric problems such as Question 2. Cartan's method concerns differential geometries that are G-structural, i.e. the geometry can be described by a G-principal sub-bundle of the coframe bundle. For example, it is a nice exercise to check that a metric g is equivalent to an $O(n)$-principal sub-bundle of the coframe bundle—the space of orthonormal coframes. More generally, any geometry that can be defined by sections of the total tensor bundle on TM (satisfying a constant stabilizer assumption) will be G-structural, with G given by the pointwise stabilizer of these sections. This encompasses most classical geometries, as well as the geometry defined by $(M, \mathcal{I})$.

In principal, given a class of G-structures, one can work through the method of equivalence and determine all of the *local* invariants of G-geometries. Furthermore, the calculation is essentially a homological one, depending only on the Lie algebra of G (its Spencer cohomology). The following table lists some examples:

Geometry	Local invariants
Riemannian	Riemannian curvature tensor
Almost Complex	Nijenhuis tensor
Conformal	Weyl tensor, Cotton tensor
EDS associated to 2nd-order PDE	Principal symbol, ?????

As noted in the table, the primary invariant one finds for 2nd-order EDS is the essentially the principal symbol[4]. What one finds is a $\mathrm{Sym}^2(\mathbb{R}^{n+1})$-bundle E over M, and the principal symbol is a (local) section σ of this bundle. I remark that, in this setup, the principal symbol is defined at each point of M, even for non-linear equations. This is because each point p of M is a 2-jet, and $\sigma(p)$ is the invariant expression of the classical symbol of the linearization of F at p. In fact, the section σ depends on a choice of coframe on M, and the real invariant is the signature of $\sigma(p)$ at each point p.

Typically one makes a choice here, corresponding to the standard division of 2nd-order PDE as elliptic, hyperbolic, parabolic, etc. So, if the section σ is everywhere positive definite, then M is elliptic in nature. If it has signature $(1, n)$ then M is hyperbolic.

I restrict attention to those 2nd-order systems that are parabolic, so that the symbol σ has constant signature $(0, n)$. These systems can be given by a definition adapted to their geometry:

Definition 3. A *(weakly) parabolic system* in $n + 1$ variables is a $2n + 2 + (n + 1)(n + 2)/2$ dimensional[5] exterior differential system $(M, \mathcal{I})$ such that any point has a neighborhood equipped with a spanning set of 1-forms

$$\theta_\varnothing, \theta_a, \omega^a, \pi_{ab} \qquad a, b = 0, \dots, n \tag{3}$$

that satisfy:

(1) The forms $\theta_\varnothing, \theta_a$ generate $\mathcal{I}$ as a differential ideal.
(2) The structure equations

$$\begin{aligned} d\theta_\varnothing &\equiv -\theta_a \wedge \omega^a \quad (\mathrm{mod}\ \theta_\varnothing) \\ d\theta_a &\equiv -\pi_{ab} \wedge \omega^b \quad (\mathrm{mod}\ \theta_\varnothing, \theta_b)\,. \end{aligned}$$

(3) The symbol relations $\pi_{ab} = \pi_{ba}$ and

$$\sum_{i=1}^{n} \pi_{ii} \equiv \theta_0 \quad (\mathrm{mod}\ \theta_\varnothing, \theta_i, \omega^a)\,.$$

The given coframing is adapted to the parabolic symbol, and is called *admissable*.

These exterior differential systems model scalar, parabolic, 2nd-order PDE—any small enough neighborhood can be given 'jet' coordinates so that the given coframing arises from the process described in the warm-up. However, there are natural parabolic systems (such as mean curvature flow) that cannot be written globally in the jets formulation.

The first two conditions exhibit $(M, \mathcal{I})$ as a 2nd-order equation. The third condition shows that the principal symbol is everywhere parabolic *and* that the sub-principal symbol is non-trivial *and* that the chosen coframing has been adapted to the symbol. In particular, it rules

[4]This is to be expected: the principal symbol is invariant under changes of coordinates, and used classically to categorize 2nd-order PDE.

[5]This is 1 less than the dimension of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$. In fact, a parabolic system can locally be defined by a hypersurface in $J^2(\mathbb{R}^{n+1}, \mathbb{R})$, as in the warmup.

out equations whose principal and sub-principal symbol have a kernel. For example, the 2-dimensional Laplace equation with 1 free parameter,

$$\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) u(x_0, x_1, x_2) = x_0, \tag{4}$$

has parabolic principal symbol, but behaves like a family of elliptic equations. Its associated parabolic system does not satisfy condition 3.

Example 2. Consider the canonical example of a parabolic differential equation, the heat equation

$$\frac{\partial u}{\partial x^0} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^i \partial x^i}.$$

The corresponding exterior differential system M is given by the submanifold $\{p_0 = \sum_{i=1}^{n} p_{ii}\}$ in $J^2(\mathbb{R}^{n+1}, \mathbb{R})$ and the ideal

$$\mathcal{I} = \{\, du - p_a \omega^a,\ dp_a - p_{ab}\omega^b\}.$$

The coframing

$$\begin{aligned} \theta_\varnothing &= du - p_a \omega^a & \qquad \pi_{ab} &= dp_{ab} \\ \theta_a &= dp_a - p_{ab}\omega^b & \qquad \omega^a &= dx^a \end{aligned}$$

restricts to an admissable coframing of M, making it into a parabolic system.

Note that condition 3 follows immediately from

$$dp_0 = \sum_{i=1}^{n} dp_{ii}.$$

In fact, the stronger statement

$$\sum_{i=1}^{n} \pi_{ii} \equiv \theta_0 \pmod{\omega^a} \tag{5}$$

holds in this case.

Returning to the definition of a parabolic system, observe that an admissable coframing is not unique. In fact, there is a Lie sub-group G of $\mathrm{GL}(\mathbb{R}^{2n+2+(n+1)(n+2)/2})$ so that any two adapted coframes at each point differ by multiplication with an element $g \in G$. More precisely, each coframing $(\theta_\varnothing, \theta_a, \omega^a, \pi_{ab})$ defines for each point p in its domain a linear map

$$u\colon T_pM \to \mathbb{R}^{2n+2+(n+1)(n+2)/2},$$

and the diagram

$$\begin{array}{ccc} & & \mathbb{R}^{2n+2+(n+1)(n+2)/2} \\ & \overset{u}{\nearrow} & \Big\downarrow g \\ T_pM & & \\ & \underset{\tilde{u}}{\searrow} & \\ & & \mathbb{R}^{2n+2+(n+1)(n+2)/2} \end{array}$$

commutes. The group G (which can of course be described explicitly) is the space of pointwise symmetries of the geometry defined by a parabolic system. Parabolic systems are equivalent to certain G-structures with this group G.

One can continue to apply Cartan's method, now to the geometry of parabolic systems. The upshot is 2 new classes of local invariants, the Monge-Ampère invariants and the Goursat invariants.

The Monge-Ampère invariants measure how far a second order equation is from being Monge-Ampère, where, by definition, a 2^{nd}-order Monge-Ampère equation is one of the form

$$F(x^a, u, p_a, p_{ab}) = \sum_{|I|=|J|} A_{I,J}(x^a, u, p_a) H_{I,J} = 0,$$

where the I, J range over subsets of $\{0, \ldots, n\}$ and $H_{I,J}$ stands for the (I, J) minor determinant of the hessian matrix

$$H = \left(\frac{\partial^2 u}{\partial x^a \partial x^b} \right).$$

In my thesis I prove the following theorem in arbitrary dimension, which was previously known by Bryant-Griffiths [1] for $n + 1 = 2$ variables and Clelland [2] for $n + 1 = 3$ variables.

Theorem 1 (McMillan). *Let F be a 2^{nd}-order parabolic PDE and $(M, \mathcal{I})$ its associated parabolic system. If the Monge-Ampère invariants of M vanish identically, then F is a Monge-Ampère equation.*

On the other hand, the Goursat invariants measures how far a parabolic system is from being evolutionary, as in the following theorem.

Theorem 2 (McMillan). *Let F be a 2^{nd}-order parabolic PDE and $(M, \mathcal{I})$ its associated parabolic system. The Goursat invariants vanish (which requires that certain Monge-Ampère invariants also vanish) if and only if F can be written in evolutionary form:*

$$F(x^a, u, p_a, p_{ab}) = p_0 - F_1(x^a, u, p_i, p_{ij})$$

where $i, j = 1 \ldots n$.

The proof furthermore gives explicit coordinates for which F is in this evolutionary form. Indeed, if the Goursat invariants vanish, then there is an admissable coframing of the form (3) so that ω^0 is closed. Locally, there is a function t so that $\omega^0 = dt$, and this function provides the 'time' foliation that defines the evolutionary form of F.

One can now proceed to define more refined invariants, which depend on the values of the Monge-Ampère and Goursat invariants. But the Monge-Ampère and Goursat invariants are already enough to begin to get a good understanding of the behavior of conservation laws of parabolic systems, which I turn to now.

The Conservation Laws of Parabolic Systems

First, I recall the general definition of a 0^{th}-order conservation law for an exterior differential system.

Definition 4. Given an exterior differential system $(M, \mathcal{I})$ whose maximal integral manifolds are $n+1$-dimensional, a 0^{th}-order conservation law is an n-form Ψ for which $d\Psi \in \mathcal{I}$.

The idea is that the form $d\Psi$ will vanish on any integral manifold Σ, so by Stokes theorem

$$\int_{\partial\Sigma} \Psi = \int_{\Sigma} d\Psi = 0.$$

For example, consider the heat equation defined above. It is not difficult to check, using equation (5), that the form

$$\Psi = p_i \omega_{(i)} - u\omega_{(0)} \tag{6}$$

(where $\omega_{(a)} = \omega^0\wedge\ldots\hat{\omega}^a\ldots\wedge\omega^n$ is the ommitted index notation) is a non-trivial conservation law.

It is clear from this formulation that if Ψ is closed, or if it is already in $\mathcal{I}$, then it doesn't define an interesting conservation law, so the space of non-trivial 0$^{\text{th}}$-order conservation laws is defined to be the middle homology of the sequence

$$\Omega^{n-1}(M)/\mathcal{I} \xrightarrow{d} \Omega^n(M)/\mathcal{I} \xrightarrow{d} \Omega^{n+1}(M)/\mathcal{I}\,.$$

Although this is the geometrically useful definition of conservation laws, there is a more computationally useful way to define them. In fact, this sequence fits into the side of a spectral sequence. This spectral sequence defines an isomorphism between the space of conservation laws and another vector space, the space of *differentiated* conservation laws. This latter space can be computed using the spectral sequence. Instead of describing the general result, I describe the space of differentiated conservation laws for parabolic systems below.

Before stating the theorem, one more technical point: strictly speaking, the conservation laws defined here only depend on few derivatives of solutions. For example, parabolic systems are modeled on subsets of the space of 2-jets, so any conservation law can depend on at most second derivatives of solutions. To consider all conservation laws, which depend on derivatives of arbitrarily high order, we use a formal process (called prolongation) that replaces the EDS $(M,\mathcal{I})$ with a larger EDS that has the same solutions, but also sees derivatives of all order. That is, there is an EDS $(M^{(\infty)},\mathcal{I}^{(\infty)})$ equipped with a submersion

$$\pi\colon M^{(\infty)} \to M$$

so that any integral manifold Σ of M has a unique lifting to an integral manifold in $M^{(\infty)}$. Generally $M^{(\infty)}$ will be an infinite dimensional manifold. Then the space of conservation laws on $M^{(\infty)}$ gives the space of conservation laws of all order on M.

For example, a parabolic system that comes embedded into $J^2(\mathbb{R}^{n+1},\mathbb{R})$ will be replaced by a new one equipped with an embedding into the infinite jet-space $J^\infty(\mathbb{R}^{n+1},\mathbb{R})$.

Definition 5. The space of all conservation laws on an EDS M is the space of 0$^{\text{th}}$-order conservation laws on $M^{(\infty)}$.

Theorem 3 (McMillan)**.** *Let Φ be a differentiated conservation law of a paraobolic system $(M,\mathcal{I})$. There is a function A on $M^{(\infty)}$ and an n-form ψ_A so that*

$$\Phi \equiv A\sum_{i=1}^{n}\theta_i\wedge\omega_{(i)} \quad \left(\mathit{mod}\ \theta_\varnothing, \Lambda^2\,\mathcal{I}\right).$$

The function A satisfies an auxilliary differential equation determined by the local invariants of M.

Example 3. The differentiated version of the heat equation conservation law given in Equation (6) is given by

$$\Phi = d\Psi = \theta_i\wedge\omega_{(i)} - \theta_\varnothing\wedge\omega_{(0)}.$$

From the theorem, the problem of classifying all of the conservation laws of a given parabolic systems comes down to solving a differential equation on $M^{(\infty)}$. A priori, this is an infinite dimensional problem because $M^{(\infty)}$ is infinite dimensional. However, the following theorem shows that the problem is finite for parabolic Monge-Ampère systems.

Theorem 4 (McMillan)**.** *Let $(M,\mathcal{I})$ be a parabolic Monge-Ampère system and Φ a differentiated conservation law as in Theorem 3. Then the function A is the pull back along π of a function on M.*

In other words, for a parabolic Monge-Ampère system, the conservation laws are determined by solutions to a PDE defined on M, a finite dimensional manifold.

This in particular rules out any behavior like the KdV hierarchy of conservation laws: all conservation laws of MA parabolic systems occur at 0^{th} order.

References

[1] Robert L. Bryant and Phillip A. Griffiths. Characteristic cohomology of differential systems II: Conservation laws for a class of parabolic equations. *Duke Math. J.*, 78(3):531–676, 06 1995.

[2] Jeanne N. Clelland. Geometry of conservation laws for a class of parabolic partial differential equations. *Selecta Mathematica*, 3(1):1–77, 1997.